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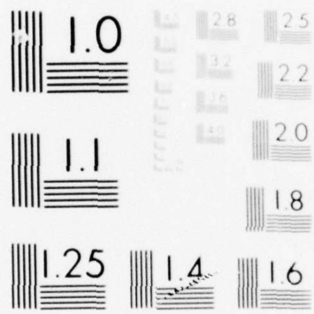

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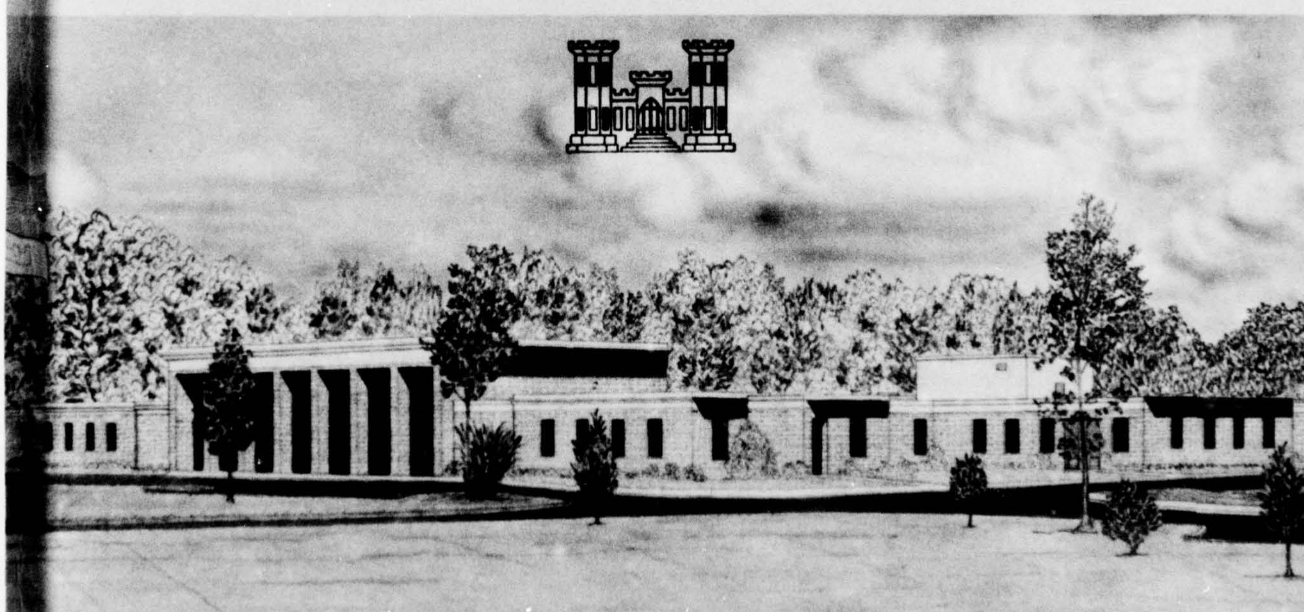
MISCELLANEOUS PAPER O-71-2

APPLICATION OF SPLINE INTERPOLATION METHODS TO ENGINEERING PROBLEMS

by

J. B. Cheek, Jr., N. Radhakrishnan, F. T. Tracy



July 1971

Conducted by U. S. Army Engineer Waterways Experiment Station, Vicksburg, Mississippi

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FOREWORD

This paper was prepared by Mr. J. B. Cheek, Jr., Dr. N. Radhakrishnan, and Mr. F. T. Tracy of the Computer Analysis Branch, Automatic Data Processing Division, U. S. Army Engineer Waterways Experiment Station (WES), for presentation to the 1971 Army Numerical Conference sponsored by the Army Mathematics Steering Committee and hosted by the Department of Defense Computer Institute. The paper was reviewed and approved for presentation and publication by the Office, Chief of Engineers, U. S. Army.

The work was conducted during the period Dec 1967 to Apr 1971 under the general supervision of Mr. D. L. Neumann, Division Chief. It is based on work done for several of the WES technical divisions and the Lower Mississippi Valley Division in connection with a number of engineering projects.

During the period in which this paper was prepared, COL Ernest D. Peixotto, CE, was Director of WES; Mr. F. R. Brown was Technical Director.

CONTENTS

SUMMARY

This paper was prepared to familiarize practicing scientists and engineers with the cubic spline interpolation technique as a possible tool in curve fitting for computer programs for which more commonly used techniques may be unsuitable or of limited value. The spline technique is compared with more common methods, specifically piece-wise linear and polynomial, and examples of applications of the technique to engineering problems are presented. Appendix A contains the mathematical derivation of the equations defining the spline function, and Appendix B contains a compact FORTRAN fitting and interpolating program.

The interpolating spline curve-fitting technique has three primary advantages:

a. The spline passes through all data points.

b. The first and second derivatives of the spline are continuous at all points.

c. The spline can be easily modified to satisfy new or additional data.

The experience of the Waterways Experiment Station (WES) in applying spline techniques to engineering problems has indicated that these advantages outweigh the additional storage and/or computation time requirements of the technique in many applications.

Since the spline function is required to pass through all data points, erratic derivative behavior may result from experimental error when the data points are numerous and closely spaced. Trial and error methods for smoothing such functions exist, but they are time consuming. WES experience has indicated that acceptable results can generally be obtained by simply selecting a more limited, more widely spaced set of the data points to which to fit the curve.

APPLICATION OF SPLINE INTERPOLATION METHODS TO ENGINEERING PROBLEMS

PART I: INTRODUCTION

Purpose and Approach

1. Many computer programs applied to engineering research and design problems must model the characteristics of nonlinear physical systems. The cubic spline offers outstanding advantages over the interpolating methods commonly used in this class of problems. It is, therefore, the purpose of this paper to point out the shortcomings of several methods and show how spline techniques may be used to advantage. This purpose is approached through discussion and examples in language and subject that are meaningful to the research and design engineer in order to bring to the practicing scientist and engineer an assurance that the cubic spline formulation offers a powerful, practical modeling and interpolating method for use in his computer codes.

Commonly Used Curve-Fitting Techniques

2. Numerical techniques and digital computers are being applied to an increasing number of civil engineering problems. This is principally due to the flexibility afforded by the numerical procedures, in that the design and research engineer can easily specify complicated boundary conditions and use nonlinear material properties. The finite difference and finite element methods are excellent examples of this growing application area. The valid description (modeling) of the nonlinear properties to the computer program is a primary consideration and is the major concern of this paper.

3. Problems of this type require that nonlinear relationships between physical quantities be available to the solution process in either functional or analytical form. This is necessary to answer questions like:

given a strain, what is the stress; or given a water elevation (stage), what is the flow (discharge). Such relationships are normally available only as data points. It is therefore necessary to use some kind of curve-fitting technique in the computer program to represent the physical system, not only at the data points, but in the intervals between data points.

4. There are many curve-fitting methods, but there is no ideal method. Consequently, a major difficulty in developing the solution process is in selecting one curve-fitting technique that is best suited to the problem at hand.

5. Two of the most commonly used curve-fitting techniques are piece-wise linear and polynomial methods, while hyperbolic function and other special purpose representations are occasionally used. There are, however, serious disadvantages in using the linear and polynomial methods. These limitations are presented in the following paragraphs to help the reader appreciate the advantages of the spline method.

Piece-wise linear interpolation

6. Disadvantages. Given a series of data points that, for engineering purposes, "exactly" represent a physical situation, there is strong motivation to use piece-wise linear interpolation between point pairs. This is especially tempting when additional data points are easy to acquire, because the nonlinearity can be modeled (to the extent required) simply by specifying additional points for the nonlinear ("curvy") regions. It appears that the only disadvantage is the additional computer memory required for the points.

7. The serious objection of discontinuous derivatives develops when modeling observed physical behavior and calculating rates of change (derivatives) from the piece-wise linear model. This difficulty caused by discontinuous derivatives is illustrated in the following example.

8. The nonlinear properties of soil in a finite element method (FEM)[1] solution may be represented with a set of stress-strain points, as shown in fig. 1. The solution process requires interpolation of a stress value for any strain value. Also required is the modulus of elasticity for those same strain values (the modulus being dY/dX, the slope of the stress-strain curve). Fig. 1 also includes the plot of modulus versus strain.

Note the abrupt change in modulus as strain progresses from region A to region B. How does this affect the FEM solution process?

9. To answer that question one must first state three characteristics of a typical nonlinear FEM process: (a) the solution procedure involves solving a series of incremental loading problems, (b) during the load cycle, the modulus of each soil element is dependent on the strain at that element, and (c) the strains increase incrementally from an initial value as additional load cycles are taken.

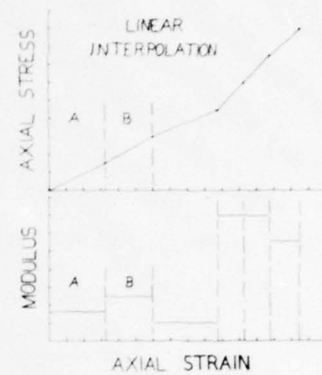

Fig. 1. Piece-wise linear stress-strain approximation

10. Thus, those elements having any value of strain in region A will have one modulus value, while those having any strain value in region B exhibit a different modulus. Consequently, as the element strain progresses from region A to B, the solution process sees an abrupt change in that element's modulus. Such behavior is not characteristic of soil modulus versus strain; i.e., the model for modulus versus strain is obviously invalid. The effect of this abrupt change is instability in the numerical process and questionable results.

11. Summary. Linear interpolation is an easy method to implement in the program. It can accurately model the observed behavior, provided a sufficient number of data points are available; but it fails completely in modeling derivatives and is wasteful of computer memory when stringent limits are placed on allowable errors.

Polynomial interpolation

12. Disadvantages. Because linear interpolation has discontinuous first derivatives, researchers have turned to higher (N^{th}) degree polynomials which have the much desired continuity of their first through $(N - 1)^{th}$ derivatives. In so doing they discovered difficulties and deficiencies associated with polynomial interpolation, some of which are outlined below:

a. Polynomials of degree N do not always pass through every data point (if there are more than N + 1 data points). This is objectionable when the observed phenomenon is such

that data points can be considered "exact" (i.e., data points whose probable error of position is small and the polynomial fit does not meet this error crition).

b. Oscillations, as many as $(N - 1)/2$, may occur between the first and last data points. Such oscillations may lie outside the range of the known behavior of the system being modeled or may incorrectly model derivative behavior.

c. It is extremely difficult to predict the overall behavior of an N^{th} degree polynomial when a data point is added, deleted, or "adjusted" (again assuming there are more than N data points). This difficulty transforms the curve-fitting process from a science to an art.

d. Although derivatives of polynomials are continuous, one cannot assume that they are representative of the physical situation being modeled.

13. Summary. Polynomial interpolation methods can be successfully applied to many nonlinear problems, but a high level of education, skill, and experience is required of the curve-fitting practitioner (the equation maker). Secondly, polynomials can never be completely trusted. New sets of data, even though they are from the same application, must be validated (plotted and examined), and most likely adjusted through a trial and error process, before the new polynomial fit can be used.

Verification of Interpolation Systems

14. It is important to note that derivative faults are easily overlooked, especially when the user restricts his evaluation of the interpolation procedure to testing its ability to reproduce a physical effect within specified error bounds. As we have demonstrated, one can have a perfectly satisfactory model of the observed behavior, while that same model is invalid in other important physical characteristics. It is, therefore, highly desirable to examine all characteristics of the physical system which the interpolating model is expected to reproduce.

PART II: CUBIC SPLINE INTERPOLATION

15. The aforementioned reasons provide motivation to look for a better, easy-to-use interpolating technique that will give smooth, predictable behavior and have continuous first derivatives. U. S. Army Engineer Waterways Experiment Station (WES) activity in several engineering areas indicates that the cubic spline[2,3] (hereinafter referred to as spline) has many advantages as an interpolating method. Several important spline characteristics are outlined below:

a. The spline passes through the data points.

b. The spline is a piece-wise cubic. The data points mark the points of transition from one set of coefficients to the next.

c. The first and second derivatives of the spline are continuous at all points.

d. The spline curve looks similar to that drawn by a french curve or a mechanical spline (more on mechanical or physical splines in paragraphs 20 and 33).

e. Adjustment of any data point affects the entire curve, but the effect is predictable.

f. The spline is uniquely defined by the X and Y coordinates of each data point and either the first or second derivative at each data point.

Mathematical Formulation

16. The mathematical spline function is a piece-wise third degree (cubic) polynomial passing through all data points and having continuous first and second derivatives.

17. The spline can be viewed as a set of cubic equations, one equation for each interval between successive data points. The coefficients of the cubic equations are such that, at any data point, the equation for the left interval will yield the same values for the first and second derivatives, respectively, as will the equation for the right interval.

18. Given a set of N data points for which the coordinates (X_i, Y_i) and second derivatives (M_i) are known for every point $(i = 1, 2, 3 \ldots N)$,

then the interpolating spline function $S(x)$ is defined as

$$S(x) = \frac{1}{6} M_i \frac{(X_{i+1} - x)^3}{X_{i+1} - X_i} + \frac{1}{6} M_{i+1} \frac{(x - X_i)^3}{X_{i+1} - X_i} + \left[Y_i - \frac{1}{6} M_i (X_{i+1} - X_i)^2\right] \frac{X_{i+1} - x}{X_{i+1} - X_i} + \left[Y_{i+1} - \frac{1}{6} M_{i+1} (X_{i+1} - X_i)^2\right] \frac{x - X_i}{X_{i+1} - X_i} \qquad (1)$$

where i is such that $X_i \leq x \leq X_{i+1}$. The first and second derivatives are

$$S'(x) = - \frac{M_i}{2} \frac{(X_{i+1} - x)^2}{X_{i+1} - X_i} + \frac{M_{i+1}}{2} \frac{(x - X_i)^2}{X_{i+1} - X_i} + \frac{Y_{i+1} - Y_i}{X_{i+1} - X_i} + \frac{1}{6} (M_i - M_{i+1})(X_{i+1} - X_i) \qquad (2)$$

$$S''(x) = M_i \left(\frac{X_{i+1} - x}{X_{i+1} - X_i}\right) + M_{i+1} \left(\frac{x - X_i}{X_{i+1} - X_i}\right) \qquad (3)$$

19. The method of evaluating the M_i coefficients is presented in Appendix A, along with a derivation of the above expressions. A compact fitting and interpolating FORTRAN program is included in Appendix B.

Discussion of Spline Characteristics

20. The spline properties discussed in paragraph 15 illuminate the advantages and disadvantages of splines. Properties a, b, c, and d combine to produce an accurate curve having continuous first and second derivatives. This, as mentioned earlier, is very useful in engineering analysis problems. Properties d and e provide a physically based insight for adjusting data to obtain a better curve. In fact, the mathematical cubic spline can be derived from the theory associated with the deflected shape of a weightless elastic beam constrained at particular points.

21. Greville[2] noted that many physical systems are correctly modeled by cubic or quadratic equations. This fact accounts, to a large extent, for the popularity of cubic splines. It also enables us to obtain a

satisfactory fit with a small number of data points when an interval of the physical phenomenon exhibits first, second, or third degree behavior.

Spline versus linear interpolation

22. Fig. 2 is a spline fit to the same set of data used in fig. 1. Note the smoothness of the curve between data points. More important is the continuous derivative behavior. This is a marked improvement over the linear interpolation model shown in fig. 1.

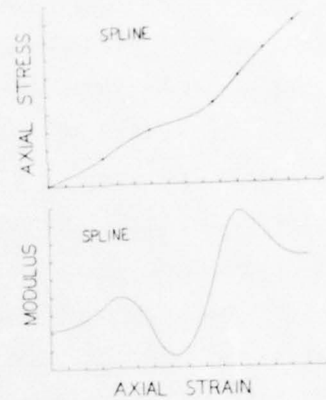

Fig. 2. Spline approximation of stress-strain relation shown fig. 1

Spline versus polynomial interpolation

23. An advantage of splines over polynomials is illustrated in fig. 3 which shows a set of 6 data points to which a fifth degree polynomial and a spline have been fit. This example shows the influence of a "bad" data value on both the spline and polynomial. Note how the spline tends to minimize the influence of the bad point P. The polynomial on the other hand is completely upset by the bad point.

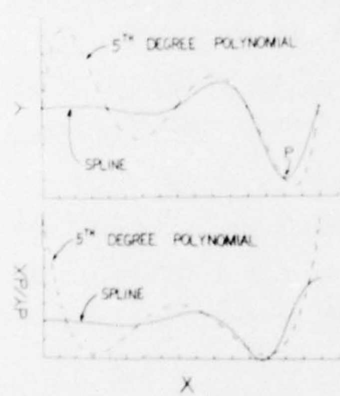

Fig. 3. General comparison of spline and polynomial fits

24. This rather extreme example is introduced not as a practical consideration but rather to illustrate the oscillating character of polynomials that is so troublesome in curve fitting. Discussion of least square polynomials is provided in the applications section, paragraphs 40-45.

Disadvantages of spline

25. Properties b and f (paragraph 15) are sometimes interpreted as disadvantages. The spline is not defined as a single expression over the entire range of the data points. Therefore, to evaluate $S(x)$ one must first find the two adjacent data points bounding x and then apply the equations of paragraph 18. This results in longer computer time than that required in polynomial fits. However, search time can be minimized by use of efficient algorithms.

26. The number of coefficients needed to define the spline (three times the number of data points) is too large for some engineering applications. This is due to the fact that the computer codes themselves (such as the finite element and finite difference codes) require vast amounts of storage to solve physical problems. For instance, the finite element analysis of soil-structure interaction problems often requires that several soil zones be considered. If the nonlinear stress-strain behavior of the soils is modeled by splines, then for each different layer of soil one must store parameters for a different spline function. Thus, the number of storage locations required by splines will be greatly increased in such problems. The same is true for finite difference solutions of ground shock problems. The engineer, quite naturally, is inclined to use the available storage for larger physical problems rather than to introduce what he may feel is an unnecessary elegance in the stress-strain interpolation routine.

<u>Summary</u>

27. Although WES experience with splines is limited, it has led to the conclusion that the advantages of the spline (continuous derivatives, smooth curve, dependability, and physical insight) far outweigh the problems of increased computer memory and somewhat longer computation times. Further, through the skill of the numerical analyst and programmer, one can exercise considerable influence in adapting the formulation and code to minimize run time or storage, a point discussed in the applications section.

PART III: APPLICATION OF SPLINES TO ENGINEERING PROBLEMS

28. This section describes the application of cubic splines to several civil engineering problems at WES. Applications of the cubic spline fitting and interpolating techniques to hydraulics problems, soil-structure interaction problems, and seepage problems are presented.

Hydraulics Problems

Rating curves

29. This application in the field of hydraulic engineering deals with modeling rating curves in a flood routing program authored by Mr. E. A. Graves of the Lower Mississippi Valley Division. Mr. Graves and Mr. J. B. Cheek, Jr., of WES are currently involved in the application of spline methods to this problem.

30. Rating curves are used to describe, at stated points (stations) on a river, the flow (discharge) characteristics as a function of the height of the water in the river (stage). The flood routing program produces very satisfactory results, but a great deal of difficulty is experienced in obtaining polynomial fits to the rating curves that are required as input to the program. The following figures and discussion illustrate the improvement brought by the spline interpolation method to the rating curve model.

31. Spline fit to 20 points. Fig. 4 shows 20 data points of a rating curve with a spline fit through those points. Except for the large number of points, the spline representation was considered satisfactory until the derivative shown in fig. 5 was examined. It was characterized by an objectionable lack of smoothness.

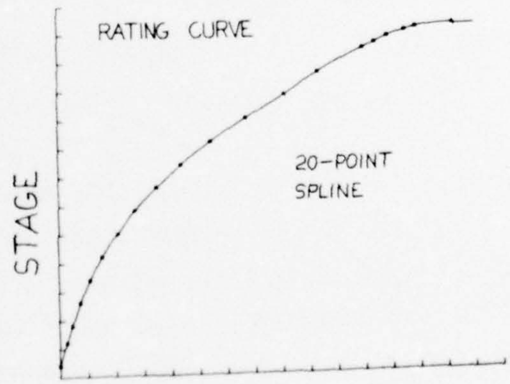

Fig. 4. Spline fit of 20-point rating curve

32. Reducing number of points to improve derivative. In order to

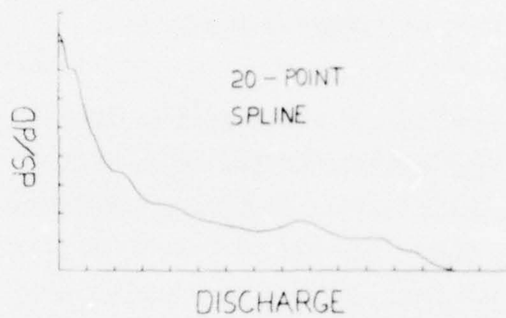

Fig. 5. Derivative for 20-point spline fit

reduce the number of data points and produce a smoother derivative curve, 6 of 20 original points were chosen and a spline was fitted to them. This is illustrated in fig. 6 along with the original 20-point curve. The 6-point curve does not match the original to the desired degree. It seems desirable to add a few more points. In a situation such as this, the physical insight to the spline has its impact.

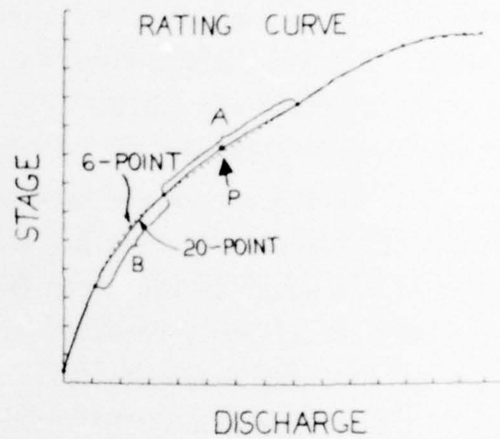

Fig. 6. Comparison of 6- and 20-point spline fits

33. As mentioned previously, the mathematical spline formulation also describes the deflection of a weightless linear elastic beam (a physical spline) which is simply constrained at the data points. This means that we can visualize the mathematical spline fit as that shape assumed by a thin strip of steel which is constrained on the paper by a straight pin at each data point. With this concept in mind, the effect of adding point P (fig. 6) to the 6-point data set can be easily predicted. The new point P would bend the spring closer to the original curve along A and would also cause a downward movement along B.

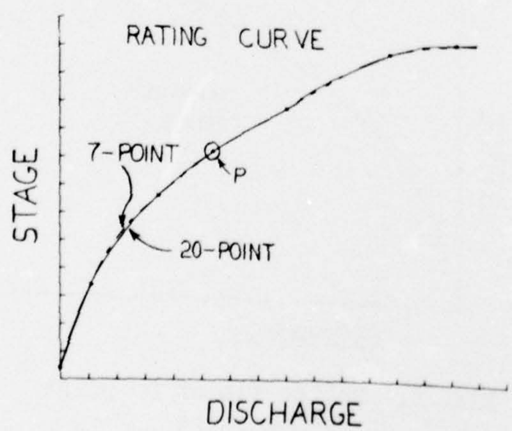

Fig. 7. Comparison of 7- and 20-point spline fits

34. Fig. 7, which shows the 7-point spline fit along with the original 20-point fit, shows the effect of adding point P. Note how

nicely the 7-point fit follows the lower portion of the curve. By some further adjustments, the upper portion of the curve could be improved, but it is not necessary for the present purpose since the point has been made, i.e., insight into the physical character of the mathematical spline makes the curve-fitting process easy.

35. The derivative for the 7-point curve (fig. 8) shows much improvement in smoothness. This improvement is presumed to be due to the reduced number of points. This aspect of spline interpolation will be discussed further in paragraphs 43-45.

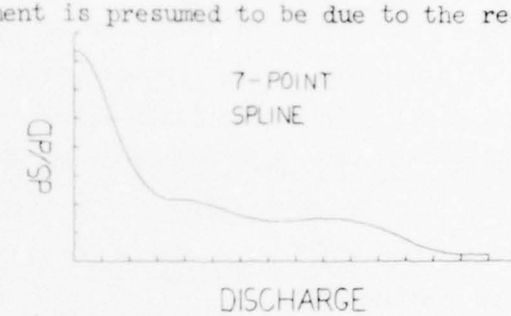

Fig. 8. Derivative for 7-point spline fit

36. Modification of spline fit. Because rating (or conveyance) curves for an alluvial river change with time, it is sometimes necessary to change them. With splines, this can be done by substituting new data points, with the assurance that the new fit will be satisfactory. This contrasts with the use of polynomials, which require many more data points and which must be carefully studied to determine if a satisfactory fit has been obtained. Also, in the application discussed, the first derivative is required, and this can be obtained directly during the spline interpolation procedure. Even in applications where the first derivative is not used, if a polynomial curve fit is to be used, it is desirable to obtain the first derivatives for use in studying the suitability of the curve to the purpose in hand.

Model storage curves

37. The success with modeling rating curves soon led to experimentation with modeling storage curves. This flood routing application has severe computer memory restrictions on it, so extra steps were taken to reduce the coefficients stored. This was accomplished by retaining only the coordinates of the points and beam moment of each data point. This is not normally done because the two other required coefficients for each point, developed during the spline fit process, must be thrown out and recalculated for the two bounding points during the interpolation computation. However, by so doing, it is possible to use the spline in a

restricted memory environment, paying for the modification with increased interpolating computation time.

38. The formulation also provides linear extrapolation for values beyond the data, using the slope and coordinates of the nearest end data point. (A tabulation of this program is presented in Appendix B.) This modification gives an extra advantage when used in a converging trial and error procedure, the advantage being that the first few trials may be outside the range of the physical data. The spline formulation makes it easy to assure that extrapolated values will aid convergence of the trial and error process. Such assurance is obtained only through additional programming in polynomial fits. It is a sufficiently difficult task to tame the polynomial within the bounds of the data without bothering with it beyond the physical data.

Soil-Structure Interaction Studies

39. In soil-structure interaction problems using the finite difference or finite element computer codes, it is necessary to represent the nonlinear stress-strain behavior of the soils. This is normally input as sets of coefficients for polynomials, one set for each soil. Using this approach, it is possible to compute the shear and bulk moduli of the material at any stress or strain level. The typical solution procedure requires the computation of the first derivative of the curve in addition to evaluating functional values from the curve itself. Spline fits are ideally suited to represent the curves because the fit is smooth, the derivatives are continuous, and only an intermediate skill level is needed to obtain a satisfactory fit to the data.

Spline versus polynomial fit

40. The 89 data points shown in fig. 9 are an example of those modeled in programs of this type. The points were computed directly from measurements taken during a test of a soil specimen. Both a spline fit and fifth degree least-squares polynomial fit are shown. The two points to the left of the pressure axis were added to force the polynomial to pass through the $(x=0, y=0)$ point.

41. Judging only from fig. 9, one could easily conclude that the spline fit is superior to the polynomial. Such a conclusion is seen to be unjustified when the spline and polynomial derivative plots in fig. 10 are considered. Incorrect derivative behavior of the polynomial (based on known characteristics of the soil) is evidenced by its negative trend near $x = 0$. However, when compared with the wild performance of the spline's derivative, the polynomial must be considered superior.

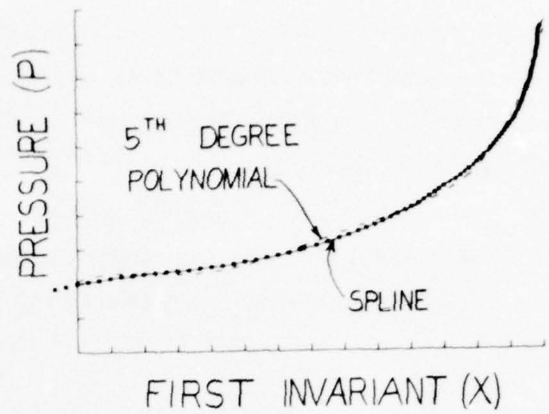

Fig. 9. Comparison of 89-point spline and polynomial fits of soil-structure interaction data

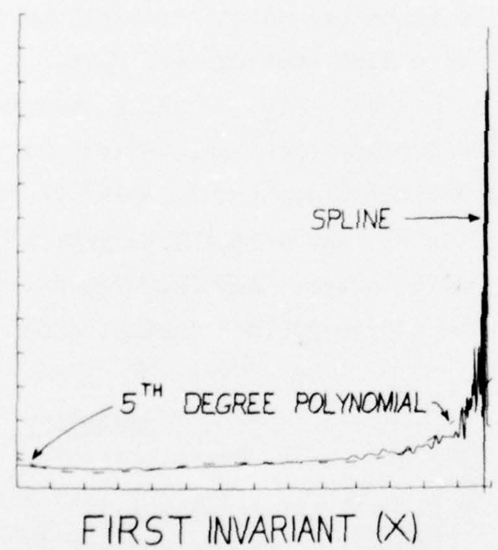

Fig. 10. Comparison of spline and polynomial derivatives for 89-point fits

Reducing number of points to improve derivative

42. Does the physical effect being modeled require all 89 points for its correct definition? Inspection of fig. 9 indicates this is unlikely. The effect on the spline of the errors in the data, small though they may be, must be considered. Note that the effect of any data error is magnified as the points are brought closer together. The large changes in slope shown in fig. 10 indicate that the spline is being subjected to many torques and countertorques that are principally due to the data error and the close spacing of the points.

43. There are smoothing spline methods that overcome this problem (by not passing through all data points). Unfortunately they require repeated trials and evaluations to obtain a satisfactory value for the

smoothing parameter. From a practical standpoint, it is generally preferable and less time consuming to select a few data points at the outset rather than to make repeated runs in search of a good parameter for each new set of data. (This is not to imply that one method is superior to the other, rather that interpolating splines can economically be made to function satisfactorily for many scientific-engineering applications.)

44. Fig. 11 shows spline and polynomial fits to 16 of the original 89 points of fig. 9. Note that the points shown are not the result of several trials to obtain the best set. They were simply selected so that the x values were about an inch apart on the original drawing, a procedure which certainly cannot be called "tuning" the data.

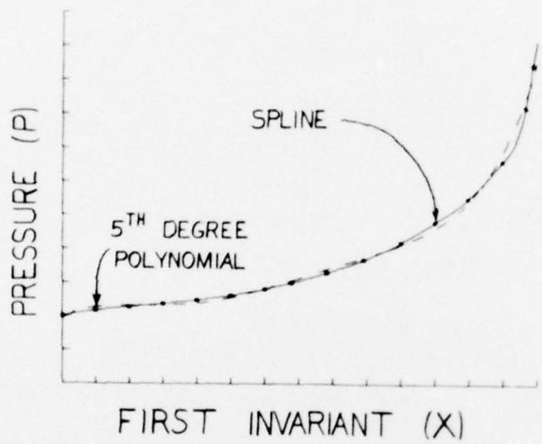

Fig. 11. Comparisons of 16-point spline and polynomial fits

45. Fig. 12 shows the vastly improved spline derivative plot which now appears superior to both the 89- and 16-point polynomial derivatives. Any improvements required can be obtained with the aid of the aforementioned insight about the physical equivalent of the cubic spline (paragraph 33).

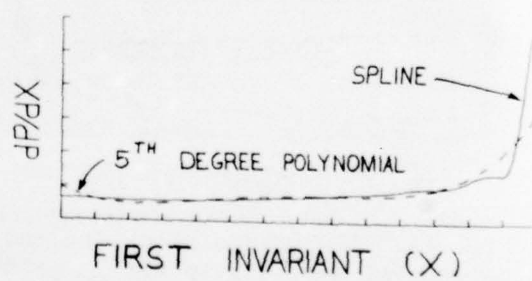

Fig. 12. Comparison of spline and polynomial derivatives for 16-point fits

Summary

46. The authors have been directly involved in three soil-structure interaction projects at WES:

a. R. E. Walker and Cheek (March 1970) on a dynamic nonlinear elastic finite element method stress analysis program.

b. J. L. Kirkland and Cheek (April 1970) on a nonlinear elastic incremental loading soil-structure interaction program.

c. PFC B. Phillips and Radhakrishnan (1970) on a finite difference solution of nonlinear elastic analysis of ground motion.

This experience has indicated that the addition of splines to soil-structure interaction codes is worthwhile. However, it cannot be stated with certainty that the computed results are better, although there are signs of improvement in the stability of the procedure (such as a reduction in strain energy growth), since no known correct solution with which to compare the results exists. It is felt that splines have removed some obvious errors in modeling nonlinear characteristics and have, at the least, allowed us to turn our research to other problems.

Steady-State Seepage Problems

Confined seepage

47. Producing a contour map from a set of data points

$$X_i, Y_i, Z_i \quad ; \quad i = 1, 2, 3 \ldots N$$

has not in general been satisfactorily accomplished. A special case of this problem where the data points occur on a rectangular grid has, however, been solved using splines. Fig. 13 shows the data configuration.

48. To construct a contour map, a surface must first be fitted to the data points. In this section, a method which fits a simple bicubic spline[4] to the data is described, and an application to confined steady-state seepage under a weir is discussed.

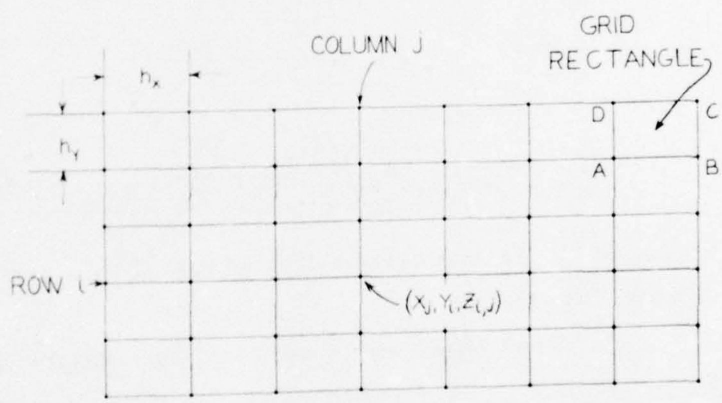

Fig. 13. Data configuration for spline contouring

49. The first part of the procedure is to fit a cubic spline along each row and each column of the data points

$$X_j, Y_i, Z_{i,j} \quad ; \quad i = 1, 2, 3 \ldots i_{max} \quad ; \quad j = 1, 2, 3 \ldots j_{max}$$

where

$$X_j = (j - 1)h_x$$
$$Y_i = (i - 1)h_y$$
$$Z_{i,j} = Z \text{ value corresponding to } (X_j, Y_i)$$

The cubic spline as defined on the j^{th} interval of the i^{th} row from property f of paragraph 15 is

$$S_i^j(x) = \frac{M_{i,j}^{(H)}}{6h_x}\left(X_{j+1} - x\right)^3 + \frac{M_{i,j+1}^{(H)}}{6h_x}\left(x - X_j\right)^3$$

$$+ \left(Z_{i,j} - \frac{1}{6} M_{i,j}^{(H)} h_x^2\right)\frac{X_{j+1} - x}{h_x} + \left(Z_{i,j+1} - \frac{1}{6} M_{i,j+1}^{(H)} h_x^2\right)\frac{x - X_j}{h_x} \qquad (4)$$

where $M_{i,j}^{(H)}$ is the second derivative of the horizontal spline passing through (i,j). Similarly, the spline as defined on the i^{th} interval of the j^{th} column is

$$S_j^i(y) = \frac{M_{i,j}^{(V)}}{6h_y}\left(Y_{i+1} - y\right)^3 + \frac{M_{i+1,j}^{(V)}}{6h_y}\left(y - Y_i\right)^3$$

$$+ \left(Z_{i,j} - \frac{1}{6} M_{i,j}^{(V)} h_y^2\right)\left(\frac{Y_{i+1} - y}{h_y}\right) + \left(Z_{i+1,j} - \frac{1}{6} M_{i+1,j}^{(V)} h_y^2\right)\left(\frac{y - Y_i}{h_y}\right) \qquad (5)$$

where $M_{i,j}^{(V)}$ is the second derivative of the vertical spline passing through (i,j).

50. From these cubic splines, the simple bicubic spline is generated. In the region (grid rectangle)

$$X_j \leq x \leq X_{j+1}$$

$$Y_i \leq y \leq Y_{i+1}$$

$$S^{i,j}(x,y) = \frac{1}{2} S^i_j(y) \frac{X_{j+1} - x}{h_x} + \frac{1}{2} S^i_{j+1}(y) \frac{x - X_j}{h_x}$$

$$+ \frac{1}{2} S^j_i(x) \frac{Y_{i+1} - y}{h_y} + \frac{1}{2} S^j_{i+1}(x) \frac{y - Y_i}{h_x} \quad (6)$$

where $S^{i,j}(x,y)$ is the bicubic spline as defined in the grid rectangle whose lower left-hand coordinates are (X_j, Y_i). From this bicubic spline, the actual construction of the contour lines can be accomplished as explained in reference 5.

51. A general purpose contouring program, documented in reference 4, which uses the above method, has been applied to several problems. One of the most important of these is producing equipotential lines from steady-state confined seepage under a weir. The problem is illustrated in fig. 14. It consists of a weir resting on a pervious homogeneous region underlain by an impervious base. Seepage occurs through the pervious medium because of the head differential between its two ends. This seepage produces uplift pressure on the base of the weir, and the engineer is *interested* in

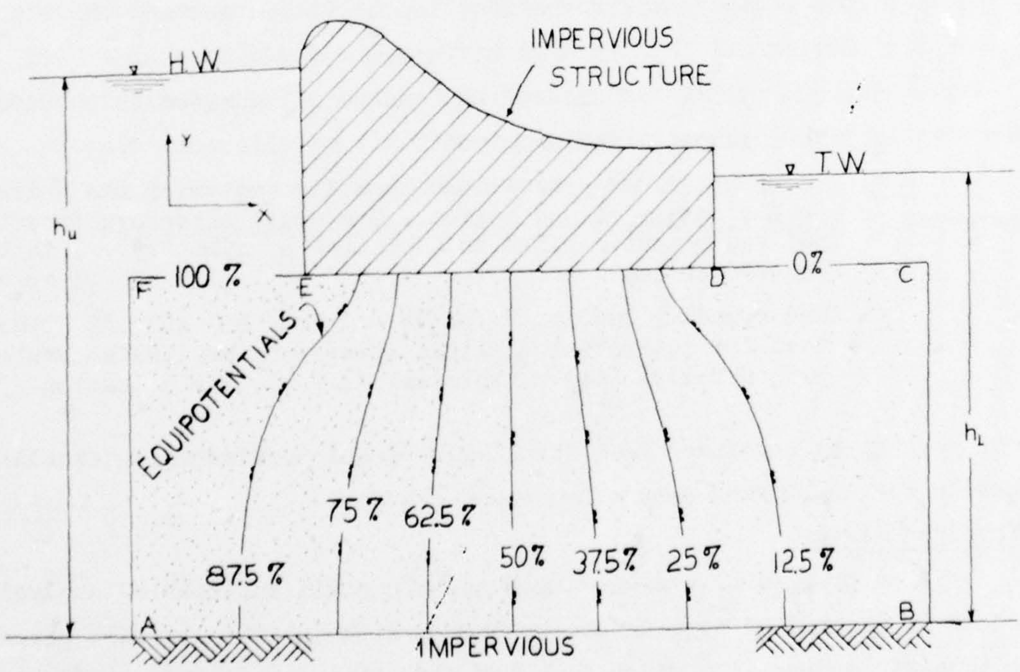

Fig. 14. Steady-state seepage under a weir

the distribution of pressures in the pervious medium. The results are normally expressed as contours of equal pressures or potentials called equipotentials.

52. Region ABCF in fig. 14 is divided into a grid system, and a finite difference solution to the governing partial differential equation (Laplace's equation)

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$$

with boundary conditions

$h = h_U$ on AFE

$h = h_L$ on DCB

$\frac{\partial h}{\partial y} = 0$ on ED and AB (ED and AB are flow lines. There is no flow across these boundaries.)

is obtained.[6] The output from this solution is used as input to the contouring program to obtain the equipotential lines. The results are shown in fig. 14. The equipotentials are labelled as the percentage $\left[(h_e - h_L)/(h_U - h_L)\right] \times 100$, where h_e is a given equipotential.

53. The quality of the contour map can be illustrated in three ways:

a. The contour lines are smooth.

b. The problem was set up such that the center of the weir was in the center of the region. One would therefore expect the 50% equipotential line to be a line of symmetry. This is indeed the case.

c. The boundary condition $\partial h/\partial y = 0$ on ED and AB requires that the equipotential lines intersect the line segments orthogonally. Again, this is, in fact, the situation.

54. It is concluded that the simple bicubic spline is an excellent function for contouring data established on a grid.

Unconfined seepage

55. A problem of classic importance in civil engineering analysis is that of unconfined flow in an earth dam, with special interest given to the phreatic surface,[7,8] which is a top flow line across which there is no flow. Another property of the phreatic surface is that the potential at

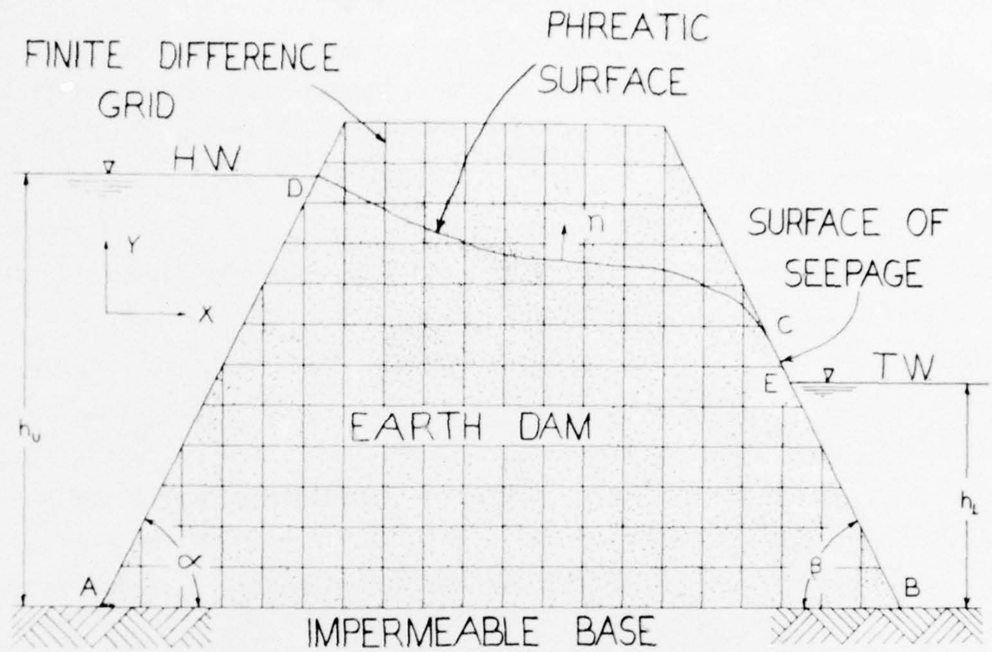

Fig. 15. Location of phreatic surface

any point on the surface is equal to the elevation head at that point.

56. This problem is illustrated in fig. 15. The phreatic surface (which is at atmospheric pressure) is itself derived from the flow equations and must generally be fixed by empirical or trial and error schemes. The trial and error procedure is described here.

57. For steady-state conditions, the governing partial differential equation is

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$$

with boundary conditions

$$\frac{\partial h}{\partial y} = 0 \quad \text{on} \quad AB$$

$$h = h_U \quad \text{on} \quad AD$$

$$h = h_L \quad \text{on} \quad BE$$

$$h = y \quad \text{on} \quad DCE$$

$$\frac{\partial h}{\partial n} = 0 \quad \text{on} \quad DC$$

The procedures used in solving this problem are as follows:

a. Make an initial guess to the phreatic surface DC .

b. Solve the problem by finite difference methods as if it were a confined seepage problem using $\partial h/\partial n = 0$ and $P/\rho g$ = constant on DC (P = pressure ; ρ = mass density of fluid ; g = acceleration due to gravity).

c. Adjust DC to satisfy $h = y$.

d. Repeat b and c until $\partial h/\partial n = 0$ and $h = y$ are satisfied on DC simultaneously.

58. The principal difficulty of this procedure is in satisfying $\partial h/\partial n = 0$ on DC .[8] This is because DC intersects the finite difference grid at irregular points, as illustrated in fig. 15. Incorporating splines into the procedure may essentially eliminate this difficulty. Step b should then be altered as follows:

a. Fit a cubic spline along DC as shown in fig. 15. Note that the points of intersection of the phreatic surface and the grid are used as data points. Use the set of equations which solves for the slopes at the data points (see Appendix A for details), and set

$$S_1 = -\cot \alpha$$

$$S_N = -\tan \beta$$

where N is the number of data points.

b. Replace $\partial h/\partial n = 0$ and $P/\rho g$ = constant on DC with the equivalent expressions

$$\left(\frac{\partial h}{\partial x}\right)_i = \frac{S_i}{1 + S_i^2}$$

$$\left(\frac{\partial h}{\partial y}\right)_i = \frac{S_i^2}{1 + S_i^2}$$

$$i = 1, 2, 3 \ldots N$$

where the S_i's are the slopes as computed by the spline fit.

c. Incorporate these simple formulas into the finite difference solution to obtain the next trial values.

PART IV: CONCLUSIONS

59. The spline interpolation technique is a valuable tool in curve fitting for computer programs for which more commonly used techniques are unsuitable or of limited value. The spline technique has three primary advantages:

- a. The spline passes through all data points.
- b. The first and second derivatives of the spline are continuous at all points.
- c. The spline can be easily modified to satisfy new or additional data.

WES experience in applying spline techniques to engineering problems has indicated that in many applications these advantages outweigh the additional storage and/or computation time requirements of the technique.

60. Since the spline function is required to pass through all data points, erratic derivative behavior may result from experimental error when the data points are numerous and closely spaced. Trial and error methods for smoothing such functions exist, but they are time consuming. WES experience has indicated that acceptable results can generally be obtained by simply selecting a more limited, more widely spaced set of the data points to which to fit the curve.

LITERATURE CITED

1. Zienkiewicz, O. C. and Cheung, Y. K., The Finite Element Method in Structural and Continuum Mechanics, McGraw-Hill, London, 1967.

2. Greville, T. N. E., "Data Fitting by Spline Functions," MRC Technical Summary Report No. 893, June 1968, Mathematics Research Center, U. S. Army, University of Wisconsin, Madison, Wisc.

3. Ahlberg, J. H., Nilson, E. N., and Walsh, J. L., Theory of Splines and Their Applications, Academic Press, 1967.

4. De Boor, C., "Bicubic Spline Interpolation," Journal of Mathematics and Physics, Vol 41, 1962, pp 212-218.

5. Long, J. T. and Tracy, F. T., "A General Purpose Contouring System," Miscellaneous Paper T-70-1, Apr 1970, U. S. Army Engineer Waterways Experiment Station, CE, Vicksburg, Miss.

6. Shaw, F. S., "Second Order Linear Partial Differential Equations," An Introduction to Relaxation Methods, Dover, New York, 1953, pp 71-84.

7. Shaw, F. S. and Southwell, R. V., "Relaxation Methods Applied to Engineering Problems; VII: Problems Relating to the Percolation of Fluids Through Porous Materials," Proceedings, Royal Society, Vol 178, No. 1, 9 May 1971, pp 1-17.

8. Finnemore, E. J. and Perry, B., "Seepage Through an Earth Dam Computed by the Relaxation Technique," Water Resources Research, Vol 4, No. 5, Oct 1968, pp 1059-1067.

APPENDIX A: EQUATIONS FOR CUBIC SPLINE

1. A cubic spline is a function whose third derivative is a step function with points of discontinuity at the data points

$$X_i, Y_i \quad ; \quad i = 1, 2, 3 \ldots N$$

where N is the number of data points. The spline function of degree three is therefore a piece-wise, continuous third degree polynomial having continuous first and second derivatives.

2. For this application the function will be further required to interpolate the data points. Hence, on the i^{th} interval

$$X_i \leq x \leq X_{i+1} \quad ; \quad i = 1, 2, 3 \ldots N - 1$$

$$S(x) = a_i + b_i x + c_i x^2 + d_i x^3 \tag{A1}$$

where a_i , b_i , c_i , and d_i are constants to be evaluated. Equation A1 can be rewritten as

$$S(x) = a_i'(x - X_i)^3 + b_i'(X_{i+1} - x)^3$$

$$+ c_i'(x - X_i) + d_i'(X_{i+1} - x) \tag{A2}$$

where a_i' , b_i' , c_i' , and d_i' are an alternate set of constants. Differentiating equation A2 twice yields

$$S''(x) = 6a_i'(x - X_i) + 6b_i'(X_{i+1} - x) \tag{A3}$$

Applying

$$S''(X_i) = M_i \tag{A4}$$

$$S''(X_{i+1}) = M_{i+1} \tag{A5}$$

where M_i is the second derivative at point (X_i, Y_i), the result is

$$a_i' = \frac{M_{i+1}}{6(X_{i+1} - X_i)} \quad \text{and} \quad b_i' = \frac{M_i}{6(X_{i+1} - X_i)} \tag{A6}$$

3. Since the spline must interpolate the data points

$$S(X_i) = Y_i \tag{A7}$$

$$S(X_{i+1}) = Y_{i+1} \tag{A8}$$

Substituting these into equation A2 produces

$$c_i' = \left[Y_{i+1} - a_i'\left(X_{i+1} - X_i\right)^3\right]\left(\frac{1}{X_{i+1} - X_i}\right)$$

$$= \left[Y_{i+1} - \frac{M_{i+1}}{6}\left(X_{i+1} - X_i\right)^2\right]\left(\frac{1}{X_{i+1} - X_i}\right) \tag{A9}$$

$$d_i' = \left[Y_i - b_i'\left(X_{i+1} - X_i\right)^3\right]\left(\frac{1}{X_{i+1} - X_i}\right)$$

$$= \left[Y_i - \frac{M_i}{6}\left(X_{i+1} - X_i\right)^2\right]\left(\frac{1}{X_{i+1} - X_i}\right) \tag{A10}$$

Collecting terms

$$S(x) = \frac{1}{6} M_1 \frac{\left(X_{i+1} - x\right)^3}{X_{i+1} - X_i} + \frac{1}{6} M_{i+1} \frac{\left(x - X_i\right)^3}{X_{i+1} - X_i}$$

$$+ \left[Y_i - \frac{1}{6} M_i\left(X_{i+1} - X_i\right)^2\right] \frac{X_{i+1} - x}{X_{i+1} - X_i}$$

$$+ \left[Y_{i+1} - \frac{1}{6} M_{i+1}\left(X_{i+1} - X_i\right)^2\right] \frac{\left(x - X_i\right)}{X_{i+1} - X_i} \tag{A11}$$

Differentiating

$$S'(x) = -\frac{M_i}{2}\frac{\left(X_{i+1} - x\right)^2}{X_{i+1} - X_i} + \frac{M_{i+1}}{2}\frac{\left(x - X_i\right)^2}{X_{i+1} - X_i}$$

$$+ \frac{Y_{i+1} - Y_i}{X_{i+1} - X_i} + \frac{1}{6}(M_i - M_{i+1})(X_{i+1} - X_i) \qquad \text{(A12)}$$

$$S''(x) = M_i\left(\frac{X_{i+1} - x}{X_{i+1} - X_i}\right) + M_{i+1}\left(\frac{x - X_i}{X_{i+1} - X_i}\right) \qquad \text{(A13)}$$

$$X_i \le x \le X_{i+1} \quad ; \quad i = 1, 2, 3 \ldots N - 1$$

4. The M_i's are evaluated by satisfying continuity of the first derivative at the data points

$$X_i, Y_i \quad ; \quad i = 2, 3, 4 \ldots N - 1$$

and specifying M_1 and M_N. At point i

$$S'(X_i-) = S'(X_i+)$$

Applying equation A12

$$\frac{M_i}{2}(X_i - X_{i-1}) + \frac{Y_i - Y_{i-1}}{X_i - X_{i-1}} + \frac{1}{6}(M_{i-1} - M_i)(X_i - X_{i-1})$$

$$= -\frac{M_i}{2}(X_{i+1} - X_i) + \frac{Y_{i+1} - Y_i}{X_{i+1} - X_i} + \frac{1}{6}(M_i - M_{i+1})(X_{i+1} - X_i) \qquad \text{(A14)}$$

Collecting terms

$$M_{i-1}(X_i - X_{i-1}) + 2M_i(X_{i+1} - X_{i-1}) + M_{i+1}(X_{i+1} - X_i) = 6\left(\frac{Y_{i+1} - Y_i}{X_{i+1} - X_i} + \frac{Y_i - Y_{i-1}}{X_i - X_{i-1}}\right) \quad \text{(A15)}$$

$$i = 2, 3, 4 \ldots N - 1$$

A good choice for M_1 and M_N is

$$M_1 = M_N = 0 \quad \text{(A16)}$$

This set of simultaneous linear equations can be solved for the M_i's .

5. It is sometimes more desirable to solve for first derivatives at the data points rather than second derivatives. This set of simultaneous equations is derived as follows. Satisfying

$$S'(X_i) = T_i \quad \text{(A17)}$$

$$S'(X_{i+1}) = T_{i+1} \quad \text{(A18)}$$

where T_i is the first derivative at (X_i, Y_i), equation A12 becomes

$$T_i = -\frac{M_i}{2}(X_{i+1} - X_i) + \frac{Y_{i+1} - Y_i}{X_{i+1} - X_i} + \frac{1}{6}(M_i - M_{i+1})(X_{i+1} - X_i) \quad \text{(A19)}$$

$$T_{i+1} = \frac{M_{i+1}}{2}(X_{i+1} - X_i) + \frac{Y_{i+1} - Y_i}{X_{i+1} - X_i} + \frac{1}{6}(M_i - M_{i+1})(X_{i+1} - X_i) \quad \text{(A20)}$$

Collecting terms

$$T_i = \left(-\frac{M_i}{3} - \frac{M_{i+1}}{6}\right)(X_{i+1} - X_i) + \frac{Y_{i+1} - Y_i}{X_{i+1} - X_i} \quad \text{(A21)}$$

$$T_{i+1} = \left(\frac{M_i}{6} + \frac{M_{i+1}}{3}\right)(X_{i+1} - X_i) + \frac{Y_{i+1} - Y_i}{X_{i+1} - X_i} \quad \text{(A22)}$$

Solving for M_i and M_{i+1}

$$M_i = \left(3\,\frac{Y_{i+1} - Y_i}{X_{i+1} - X_i} - 2T_i - T_{i+1}\right)\left(\frac{2}{X_{i+1} - X_i}\right) \tag{A23}$$

$$M_{i+1} = \left(T_i + 2T_{i+1} - 3\,\frac{Y_{i+1} - Y_i}{X_{i+1} - X_i}\right)\left(\frac{2}{X_{i+1} - X_i}\right) \tag{A24}$$

Requiring continuity of the second derivative at data points $i = 2, 3, 4 \ldots N - 1$

$$S''(X_i+) = S''(X_i-) \tag{A25}$$

Applying equations A12 and A24

$$\left(3\,\frac{Y_{i+1} - Y_i}{X_{i+1} - X_i} - 2T_i - T_{i+1}\right)\left(\frac{2}{X_{i+1} - X_i}\right) = \left(T_{i-1} + 2T_i - 3\,\frac{Y_i - Y_{i-1}}{X_i - X_{i-1}}\right)\left(\frac{2}{X_i - X_{i-1}}\right) \tag{A26}$$

Or

$$(X_{i+1} - X_i)T_{i-1} + 2(X_{i+1} - X_{i-1})T_i + (X_i - X_{i-1})T_{i+1} = 3\left[\left(\frac{X_{i+1} - X_i}{X_i - X_{i-1}}\right)(Y_i - Y_{i-1}) + \left(\frac{x - X_{i-1}}{X_{i+1} - X_i}\right)(Y_{i+1} - Y_i)\right] \tag{A27}$$

$$i = 2, 3, 4 \ldots N - 1$$

$M_1 = M_N = 0$ becomes

$$2T_1 + T_2 = 3\,\frac{Y_2 - Y_1}{X_2 - X_1} \tag{A28}$$

$$T_{N-1} + 2T_N = 3\,\frac{Y_N - Y_{N-1}}{X_N - X_{N-1}} \tag{A29}$$

This system may be solved for the first derivatives at the data points.

APPENDIX B: SPLINE FITTING AND INTERPOLATING SUBROUTINES

1. This appendix contains listings of the two FORTRAN language subroutines, SPLINE (table B1[2]) and SPLINT (table B2). The following sections describe the use of the two routines.

Subroutine SPLINE

2. The SPLINE subroutine is used to fit a cubic spline to a set of (x,y) data. That is, it calculates the moments at each interior data point, assigning zero to the moments at the end points. Note that only three arrays are required for each spline, an x , a y , and a moment array. The spline fit is accomplished by a call statement in the user's program of the form:

```
CALL SPLINE (A1, A2, N3, A4)
```

where the arguments A1, A2, N3, and A4 are as follows:

Argument	Purpose
A1	Names the independent variable array (x).
A2	Names the dependent variable array (y).
N3	Specifies the number of (x,y) points. N3 must be in integer form.
A4	Names the array into which SPLINE will store the calculated moments.

3. Note that the user must specify the size of arrays A1, A2, and A4 through DIMENSION or COMMON statements. Several splines may be fit and saved for subsequent use by making successive calls to SPLINE using different names for arguments A1, A2, and A4 and indicating the number of points through argument N3.

4. Example: Given two sets of (x,y) data, fit a spline to each set of data.

5. Solution: Let one set of data be in arrays X1 and Y1 having N1 points. Let the other set of data be in X2, Y2 having N2 points. The following FORTRAN statements are required, assuming that there are no more than 30 points in either data set.

```
        DIMENSION X1(30), Y1(30), C1(30), X2(30), Y2(30), C2(30)
                                  :
                   Statements to input the two data
                   sets and specify their size in
                   N1 and N2. (Note that C1 and C2
                   need not be set to any specific
                   value.)
                                  :
        CALL SPLINE (X1, Y1, N1, C1)
        CALL SPLINE (X2, Y2, N2, C2)
```

6. At this point array C1 will contain N1 moments, and array C2 will contain N2 moments; the first and last moment in each array will be zero.

7. Note that N1 need not equal N2 but neither may be less than 2 nor greater than the maximum size specified for the associated arrays (30 in this example).

Subroutine SPLINT

8. Subroutine SPLINT (SPLine INTerpolate) operates on a cubic spline defined by the (x,y) coordinates of N points and the moment at each of those points. It calculates values of y and y' for any value XX of the independent variable x . Should the value XX lie beyond the range of data defined by the points, this program will extrapolate linearly from the first (or last) using the slope of the spline at that end point.

9. Interpolating is accomplished by a call statement of the form:

```
CALL SPLINT (A1, A2, A3, A4, A5, A6, N7)
```

where the arguments A1-A6 and N7 are as follows:

Argument	Purpose
A1	Specifies the value XX of the independent variable x for which y and y' are desired.
A2	Receives the value of y at XX computed by SPLINT.
A3	Receives the value of y' at XX computed by SPLINT.
A4	Names the independent variable array of the spline.
A5	Names the dependent variable array of the spline.
A6	Names the moment array of the spline.
N7	Specifies the number of (x,y) points that define the spline. N7 must be in integer form.

10. Note that arrays A4, A5, and A6 must contain N7 values each of x , y , and moment, respectively; i.e., they contain the spline defining data. Argument A1 is an input argument for XX , while A2 and A3 receive the values for y and y' calculated by SPLINT.

11. Two of the variables in subroutine SPLINT may be useful in some applications. Variable FPPXX contains the value for y" . Variable M indicates whether the computation was an interpolation, in which case M is zero, or an extrapolation, in which case M is -1. A reduction in run time would likely result from incorporating a more sophisticated search procedure than that used (statement numbers 100 through 140 in table B2).

12. Example: Assuming that the steps outlined in the example for subroutine SPLINE have been taken, the following statements would calculate y values (YY) and y' (YPRIME) at XX = 36.49 from the second set of spline data.

```
XX = 36.49

CALL SPLINT (XX, YY, YPRIME, X2, Y2, C2, N2)
```

Test Program

13. Table B3 shows a simple test program and the ten interpolated values computed using the GE 430 Time Sharing system at the U. S. Army Engineer Waterways Experiment Station.

Table B1

Subroutine SPLINE

```
      SUBROUTINE SPLINE (X, ZY, N, S2)
C     SPLINE FITTING SUBROUTINE ADAPTED FROM
C     WORK BY GREVILLE, U S ARMY MATH. RESEARCH CENT.
C     UNIV OF WISCONSON, T S REPORT \893,
C     JUNE 1968.
      DIMENSION X(1), ZY(1), S2(1)
     DATA EPSLN /1.E-6/
      N1 = N - 1
      ASSIGN 110 TO ISW
      DO 130 I = 1, N1
      H = X(I + 1) - X(I)
      DLY = (ZY(I + 1) - ZY(I)) / H
      GO TO ISW, (110, 100)
100   H2ZZZ = HL + H
      S2(I) = 2. * (DLY - YL) / H2ZZZ
      GO TO 120
110   ASSIGN 100 TO ISW
120   HL = H
      YL = DLY
130   CONTINUE
C
      S2(1) = 0.
      S2(N) = 0.
      OMEGA = - 1.0717968
140   ETA = 0.
      ASSIGN 170 TO ISW1
      DO 190 I = 1, N1
      H = X(I + 1) - X(I)
      DLY = (ZY(I + 1) - ZY(I)) / H
      GO TO ISW1, (170, 150)
150   H2ZZZ = HL + H
      BI = .5 * HL / H2ZZZ
      W = (BI * S2(I - 1) + (.5 - BI) * S2(I + 1) + S2(I) +
&      3. * (YL - DLY) / H2ZZZ) * OMEGA
      S2(I) = S2(I) + W
      Z = ABS(W)
      IF (Z - ETA) 180, 180, 160
160   ETA = Z
      BETA = S2(I) - W
      GO TO 180
170   ASSIGN 150 TO ISW1
180   HL = H
      YL = DLY
190   CONTINUE
      IF (ABS(BETA) * EPSLN - ETA) 140, 140, 200
200   CONTINUE
      RETURN
      END
```

Table B2

Subroutine SPLINT

```
      SUBROUTINE SPLINT (XB, FXX, FPXX, X, ZY, S2, N)
C     SPLINE INTERPOLATING SUBROUTINE , BY JAY CHEEK
      DIMENSION X(1), ZY(1), S2(1)
      MM = 0
      XP = XB
      I = 1
      IF (XP - X(1)) 100, 170, 110
100   MM = - 1
      XP = X(1)
      GO TO 170
110   IF (XP - X(N)) 130, 150, 140
120   IF (XP - X(I)) 160, 170, 130
130   I = I + 1
      GO TO 120
140   MM = - 1
      XP = X(N)
150   I = N
160   I = I - 1
170   HT1 = XP - X(I)
      HT2 = XP - X(I + 1)
      PROD = HT1 * HT2
      DX = X(I + 1) - X(I)
      DELY = (ZY(I + 1) - ZY(I)) / DX
      S3 = (S2(I + 1) - S2(I)) / DX
      FPPXX = S2(I) + HT1 * S3
      DELSQS = (S2(I) + S2(I + 1) + FPPXX) / 6.
      FXX = ZY(I) + HT1 * DELY + PROD * DELSQS
      FPXX = DELY + (HT1 + HT2) * DELSQS + PROD * S3 / 6.
      IF (MM.EQ.0) GO TO 180
      FXX = FXX + FPXX * (XB - XP)
180   CONTINUE
      RETURN
      END

READY
```

Table B3

Test Program for SPLINE and SPLINT Subroutines

```
1010      DIMENSION X(10), Y(10), C(10)
1020C  SET THE TEST DATA.
1030      X(1) = 1.6
1040      Y(1) = 1.
1050      X(2) = 5.4
1060      Y(2) = 2.
1070      X(3) = 7.
1080      Y(3) = 1.
1090      X(4) = 8.2
1100      Y(4) = 1.
1110      NUMB = 4
1120C
1130C  FIT A SPLINE THROUGH THE X,Y DATA POINTS.
1140      CALL SPLINE (X, Y, NUMB, C)
1150C
1160C  INTERPOLATE AT INTERVALS OF ONE.
1170      PRINT 300
1180      DO 100 I=1, 10
1190      XI = I
1200      CALL SPLINT (XI, YY, YP, X, Y, C, NUMB)
1210 100  PRINT 200, XI, YY, YP
1220      STOP
1230 200  FORMAT (3X 3E20.9)
1240 300  FORMAT (//10X 1HX, 19X 1HY, 19X 7HY PRIME /)
1250      END

READY
RUN


JJJ       08:31   WES    05/19/71



          X                   Y                   Y PRIME

   0.100000000E+01     0.606953327E+00     0.655077788E+00
   0.200000000E+01     0.126029407E+01     0.642049980E+00
   0.300000000E+01     0.184263327E+01     0.495487139E+00
   0.400000000E+01     0.219698582E+01     0.186076697E+00
   0.500000000E+01     0.216050413E+01     -.286181346E+00
   0.600000000E+01     0.160917168E+01     -.727131570E+00
   0.700000000E+01     0.100000000E+01     -.338579530E+00
   0.800000000E+01     0.967082546E+00     0.155182285E+00
   0.900000000E+01     0.113543181E+01     0.169289765E+00
   0.100000000E+02     0.130472158E+01     0.169289765E+00
STOP

RUNNING TIME:    3.2 SECS   I/O TIME :     .4 SECS
```

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13. ABSTRACT
This paper was prepared to familiarize practicing scientists and engineers with the cubic spline interpolation technique as a possible tool in curve fitting for computer programs for which more commonly used techniques may be unsuitable or of limited value. The spline technique is compared with more common methods, specifically piecewise linear and polynomial, and examples of applications of the technique to engineering problems are presented. Appendix A contains the mathematical derivation of the equations defining the spline function, and Appendix B contains a compact FORTRAN fitting and interpolating program. The interpolating spline curve-fitting technique has three primary advantages: (a) the spline passes through all data points; (b) the first and second derivatives of the spline are continuous at all points; and (c) the spline can be easily modified to satisfy new or additional data. The experience of the Waterways Experiment Station (WES) in applying spline techniques to engineering problems has indicated that these advantages outweigh the additional storage and/or computation time requirements of the technique in many applications. Since the spline function is required to pass through all data points, erratic derivative behavior may result from experimental error when the data points are numerous and closely spaced. Trial and error methods for smoothing such functions exist, but they are time consuming. WES experience has indicated that acceptable results can generally be obtained by simply selecting a more limited, more widely spaced set of the data points to which to fit the curve.

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